# LEVEL ALGEBRAS THROUGH BUCHSBAUM* MANIFOLDS

UWE NAGEL*

Abstract. Stanley-Reisner rings of Buchsbaum* complexes are studied by means of their quotients modulo a linear system of parameters. The socle of these quotients is computed. Extending a recent result by Novik and Swartz for orientable homology manifolds without boundary, it is shown that modulo a part of their socle these quotients are level algebras. This provides new restrictions on the face vectors of Buchsbaum* complexes.

## 1. Introduction

This note is inspired by work by Novik and Swartz [13] and by Athanasiadis and Welker [1]. Its goal is to generalize some results of [13] and to contribute to the fruitful interaction of algebraic, combinatorial, and topological methods in order to study simplicial complexes.

Let $\Delta$ be a finite simplicial complex. Important algebraic properties of it like Cohen-Macaulayness or Buchsbaumness are defined by means of its Stanley-Reisner ring $K[\Delta]$. However, these properties turn out to be topological properties in the sense that they depend only on the homeomorphism type of the geometric realization $|\Delta|$ of $\Delta$. For example, all triangulations of a manifold (with or without boundary) are Buchsbaum, and all triangulations of a sphere or a ball are Cohen-Macaulay over each field.

In [1], Athanasiadis and Welker introduced Buchsbaum* complexes as the $(d-1)$-dimensional Buchsbaum complexes over a field $K$ such that

$$\dim_K \widetilde{H}_{d-2}(|\Delta| - p; K) = \dim_K \widetilde{H}_{d-2}(|\Delta|; K)$$

for every $p \in |\Delta|$, where $\widetilde{H}_j(|\Delta|, K)$ denotes the reduced singular homology of the geometric realization $|\Delta|$. The class of Buchsbaum* complexes includes all doubly Cohen-Macaulay complexes and all triangulations of orientable homology manifolds without boundary (see [1]).

In [12] Novik and Swartz pioneered the study of Buchsbaum complexes by investigating socles of artinian reductions of their Stanley-Reisner rings. Our first main result is an improvement of their description of the socle ([12], Theorem 2.2) for Buchsbaum* complexes. In fact, it provides a numerical characterization of such complexes.

**Theorem 1.1.** *Let $\Delta$ be a $(d-1)$-dimensional Buchsbaum simplicial complex. Then $\Delta$ is a Buchsbaum* complex if and only if, for each linear system of parameters $\boldsymbol{\ell}$ of $K[\Delta]$ and each positive integer $j$,*

$$\dim_K [\operatorname{Soc} K[\Delta]/\boldsymbol{\ell}]_j = \binom{d}{j} \beta_{j-1}(\Delta),$$

*where $\beta_j(\Delta) := \dim_K \tilde{H}_j(\Delta; K)$ is the $j$-th reduced Betti number of $\Delta$.*

* Part of the work for this paper was done while the author was sponsored by the National Security Agency under Grant Number H98230-09-1-0032.

This extends [13], Theorem 1.3 because each triangulation of an orientable $K$-homology manifold without boundary is a Buchsbaum* complex. Similarly, the following result extends [13], Theorem 1.4.

**Theorem 1.2.** *Let $\Delta$ be a $(d-1)$-dimensional Buchsbaum* complex, and let $\boldsymbol{\ell} := \ell_1, \dots, \ell_d$ be a linear system of parameters for $K[\Delta]$. Set $I := \bigoplus_{j=1}^{d-1} [\operatorname{Soc} K[\Delta]/\boldsymbol{\ell}]_j$. Then $(K[\Delta]/\boldsymbol{\ell})/I$ is a level ring of Cohen-Macaulay type $\beta_{d-1}(\Delta)$ and socle degree $d$, that is, the socle of $(K[\Delta]/\boldsymbol{\ell})/I$ is a $K$-vector space of dimension $\beta_{d-1}(\Delta)$ that is concentrated in degree $d$.*

Being a level ring provides strong restrictions on the Hilbert function of $(K[\Delta]/\boldsymbol{\ell})/I$. However, the classification of Hilbert functions of level algebras is a wide open problem, and the above result lends further motivation to studying it.

This note is organized as follows. In Section 2 we study the socle of the artinian reduction of a Buchsbaum complex. We observe that these complexes can in fact be characterized by their socle (Corollary 2.2). Moreover, we slightly improve the description of this socle given in [12], Theorem 2.2, by identifying one more piece. Combined with a result in [1], this implies Theorem 1.1.

Section 3 is devoted to the proof of Theorem 1.2. It allows us to establish results on the enumeration of the faces of Buchsbaum* complexes (Theorem 4.4) that improve the corresponding results for arbitrary Buchsbaum complexes in [12]. This is carried out in Section 4.

## 2. Socles of artinian reductions

Throughout this note $S := K[x_1, \dots, x_n]$ denotes the polynomial ring in $n$ variables over a field $K$.

Let $M = \oplus_{j \in \mathbb{Z}} [M]_j$ be a finitely generated, graded $S$-module. Its $i$-th local cohomology module with support in the maximal ideal $\mathfrak{m} := (x_1, \dots, x_n)$ is denoted by $H^i_{\mathfrak{m}}(M)$ (see, e.g., [6] and [17]. The *socle* of $M$ is the submodule

$$\operatorname{Soc} M := 0 :_M \mathfrak{m} := \{y \in M \mid \mathfrak{m} y = 0\}.$$

The module $M(i)$ is the module with the same structure as $M$, but with a shifted grading defined by $[M(i)]_j := [M]_{i+j}$. Furthermore we use $sM$ to denote the direct sum of $s$ copies of $M$. If $M$ has (Krull) dimension $d$, a sequence $\ell_1, \dots, \ell_d \in S$ of linear forms is called a *linear system of parameters* of $M$ if $M/\boldsymbol{\ell} := M/(\ell_1, \dots, \ell_d)M$ has dimension zero. In this case $M/\boldsymbol{\ell}$ is called an *artinian reduction* of $M$.

Assume now that $M$ is a Buchsbaum module. For a comprehensive introduction to the theory of Buchsbaum modules we refer to [19]. Here we only need the following facts about Buchsbaum modules:

- For all $i \neq \dim M$, $\mathfrak{m} H^i_{\mathfrak{m}}(M) = 0$;
- If $\ell \in S$ is a linear parameter of $M$, that is $\dim M/\ell M = \dim M - 1$, then also $M/\ell M$ is Buchsbaum and the kernel of the multiplication map $M(-1) \xrightarrow{\ell} M$ has Krull dimension zero.

Thus, the long exact cohomology sequence induced by the multiplication map splits into short exact sequences

$$0 \to H^i_{\mathfrak{m}}(M)(-1) \to H^i_{\mathfrak{m}}(M/\ell M) \to H^{i+1}(M) \to 0 \quad \text{if} \quad i+1 < \dim M =: d$$

and ends with

$$0 \to H^{d-1}_{\mathfrak{m}}(M)(-1) \to H^{d-1}_{\mathfrak{m}}(M/\ell M) \to H^d_{\mathfrak{m}}(M)(-1) \to H^d_{\mathfrak{m}}(M) \to 0.$$

Using the first sequence repeatedly one obtains, for every part $\ell_1, \ldots, \ell_j$ of a linear system of parameters of $M$, the isomorphism of graded modules

$$H^i_{\mathfrak{m}}(M/(\ell_1, \ldots, \ell_j)M) \cong \bigoplus_{k=0}^{j} \binom{j}{k} H^{i+k}_{\mathfrak{m}}(M)(-k) \quad \text{if } \ i < d - j.$$

In the case $i = 0$ and $j = d$, Novik and Swartz [12] established the following result on the socle of the module on the left-hand side.

**Theorem 2.1** ([12], Theorem 2.2)**.** *Let $M$ be a finitely generated graded Buchsbaum $S$-module of dimension $d$, and let $\boldsymbol{\ell} := \ell_1, \ldots, \ell_d$ be a linear system of parameters of $M$. Then*

$$\operatorname{Soc} M/\boldsymbol{\ell} \cong \left( \bigoplus_{j=0}^{d-1} \binom{d}{j} H^j_{\mathfrak{m}}(M)(-j) \right) \oplus Q(-d),$$

*where $Q$ is a graded submodule of $\operatorname{Soc} H^d_{\mathfrak{m}}(M)$.*

In this section we make two comments about this result in the case the module $M$ is a Stanley-Reisner ring.

A simplicial complex $\Delta$ on $n$ vertices is a collection of subsets of $[n] := \{1, \ldots, n\}$ that is closed under inclusion. Its Stanley-Reisner ideal is

$$I_\Delta := (x_{j_1} \cdots x_{j_k} \mid \{j_1 < \cdots < j_k\} \notin \Delta) \subset S,$$

and its Stanley-Reisner ring is $K[\Delta] := S/I_\Delta$. For each subset $F \subset [n]$, the *link of* $F$ is the subcomplex

$$\operatorname{lk}_\Delta F := \{G \in \Delta : F \cup G \in \Delta, F \cap G = \emptyset\}.$$

Note that $\operatorname{lk}_\Delta \emptyset = \Delta$.

The complex $\Delta$ is a Cohen-Macaulay or a Buchsbaum complex over $K$ if $K[\Delta]$ has the corresponding property. Alternatively, a combinatorial-topological characterization of Cohen-Macaulay complexes is due to Reisner [14]. Schenzel extended his result in [15]. A simplicial complex $\Delta$ is Buchsbaum (over $K$) if and only if $\Delta$ is pure and the link of each vertex of $\Delta$ is Cohen-Macaulay (over $K$).

Our first observation about Theorem 2.1 states that its description of the socle in fact characterizes Buchsbaum simplicial complexes.

**Corollary 2.2.** *Let $\Delta$ be a simplicial complex of dimension $d-1$, and let $\boldsymbol{\ell} := \ell_1, \ldots, \ell_d$ be a linear system of parameters of its Stanley-Reisner ring $K[\Delta]$. Then $\Delta$ is Buchsbaum if and only if*

$$\operatorname{Soc} K[\Delta]/\boldsymbol{\ell} \cong \left( \bigoplus_{j=1}^{d-1} \binom{d}{j} H^j_{\mathfrak{m}}(K[\Delta])(-j) \right) \oplus Q(-d),$$

*where $Q$ is a graded submodule of $\operatorname{Soc} H^d_{\mathfrak{m}}(K[\Delta])$.*

*Proof.* By Theorem 2.1, it is enough to show sufficiency. The formula for the socle implies that, for each $j \in \{1, \ldots, d-1\}$, the module $H^j_{\mathfrak{m}}(K[\Delta])$ is annihilated by $\mathfrak{m}$ and finitely generated, hence it is a finite-dimensional $K$-vector space. Therefore $\Delta$ must be a Buchbaum complex (see, e.g., [15] or [10]). □

Our second observation identifies a piece of the module $Q$ occurring in Theorem 2.1. In its proof we use

$$e(P) := \sup\{j \in \mathbb{Z} \mid [P]_j \neq 0\}$$

to denote the end of a graded module $P$. Note that $e(N) = -\infty$ if $N$ is trivial.

**Proposition 2.3.** *Let $\Delta$ be a Buchsbaum simplicial complex of dimension $d-1$, and let $\boldsymbol{\ell} := \ell_1, \ldots, \ell_d$ be a linear system of parameters of $K[\Delta]$. Then*

$$\operatorname{Soc} K[\Delta]/\boldsymbol{\ell} \cong \left( \bigoplus_{j=1}^{d-1} \binom{d}{j} H^j_{\mathfrak{m}}(K[\Delta])(-j) \right) \oplus ([H^d_{\mathfrak{m}}(K[\Delta])]_0 S)(-d) \oplus Q'(-d),$$

*where $[H^d_{\mathfrak{m}}(K[\Delta])]_0 S$ is the submodule of $H^d_{\mathfrak{m}}(K[\Delta])$ generated by its elements of degree zero and where $Q'$ is a graded submodule of $\operatorname{Soc} H^d_{\mathfrak{m}}(K[\Delta])$ that vanishes in all non-negative degrees.*

*Proof.* Hochster's formula (see [17], Theorem 4.1) provides that all intermediate cohomology modules $H^0_{\mathfrak{m}}(K[\Delta]), \ldots, H^{d-1}_{\mathfrak{m}}(K[\Delta])$ are concentrated in degree zero and that $H^d_{\mathfrak{m}}(K[\Delta])$ vanishes in all positive degrees.

Set $\boldsymbol{\ell}_j := \ell_1, \ldots, \ell_j$. Since $\boldsymbol{\ell}_d = \boldsymbol{\ell}$ and $H^0_{\mathfrak{m}}(K[\Delta]/\boldsymbol{\ell}_d) \cong K[\Delta]/\boldsymbol{\ell}_d$, our claim follows, once we have shown that, for all $j \in \{0, \ldots, d\}$,

$$e(H^{d-j}_{\mathfrak{m}}(K[\Delta]/\boldsymbol{\ell}_j)) \leq j \quad \text{and} \quad \dim_K [H^{d-j}_{\mathfrak{m}}(K[\Delta]/\boldsymbol{\ell}_j)]_j = \dim_K H^d_{\mathfrak{m}}(K[\Delta]).$$

Indeed, if $j = 0$, this is true. Let $j < d$. Multiplication by $l_{j+1}$ on $K[\Delta]/\boldsymbol{\ell}_j$ induces the long exact cohomology sequence

$$0 \to H^{d-j-1}_{\mathfrak{m}}(K[\Delta]/\boldsymbol{\ell}_j) \to H^{d-j-1}_{\mathfrak{m}}(K[\Delta]/\boldsymbol{\ell}_{j+1}) \to H^{d-j}_{\mathfrak{m}}(K[\Delta]/\boldsymbol{\ell}_j)(-1) \to H^{d-j}_{\mathfrak{m}}(K[\Delta]/\boldsymbol{\ell}_j).$$

Since $H^{d-j-1}_{\mathfrak{m}}(K[\Delta]/\boldsymbol{\ell}_j) \cong \bigoplus_{i=0}^{j} \binom{j}{i} H^{d-j-1+i}_{\mathfrak{m}}(K[\Delta])(-i)$ vanishes in each degree $k > j$, we get $e(H^{d-j-1}_{\mathfrak{m}}(K[\Delta]/\boldsymbol{\ell}_{j+1})) \leq j+1$ and

$$\dim_K [H^{d-j-1}_{\mathfrak{m}}(K[\Delta]/\boldsymbol{\ell}_{j+1})]_{j+1} = \dim_K [H^{d-j}_{\mathfrak{m}}(K[\Delta]/\boldsymbol{\ell}_j)]_j,$$

as required. □

**Remark 2.4.** The above argument shows that for an arbitrary, not necessarily Buchsbaum, simplicial complex $\Delta$, the module $[H^d_{\mathfrak{m}}(K[\Delta])]_0 S$ contributes to the socle of $K[\Delta]/\boldsymbol{\ell}$. This also follows from Lemma 2.3 in [2].

We now identify an instance where the module $Q'$ appearing in the previous result vanishes.

**Corollary 2.5.** *Let $\Delta$ be a Buchsbaum* simplicial complex of dimension $d-1$, and let $\boldsymbol{\ell} := \ell_1, \ldots, \ell_d$ be a linear system of parameters of $K[\Delta]$. Then*

$$\operatorname{Soc} K[\Delta]/\boldsymbol{\ell} \cong \left( \bigoplus_{j=1}^{d-1} \binom{d}{j} H^j_{\mathfrak{m}}(K[\Delta])(-j) \right) \oplus ([H^d_{\mathfrak{m}}(K[\Delta])]_0 S)(-d).$$

*Proof.* According to [1], Proposition 2.8, the socle of $H^d_{\mathfrak{m}}(K[\Delta]$ is concentrated in degree zero. Hence Proposition 2.3 gives the claim. □

The last result *proves Theorem 1.1* because $H^j_{\mathfrak{m}}(K[\Delta]) = [H^j_{\mathfrak{m}}(K[\Delta])]_0$ if $j \neq d$ and $\dim_K [H^j_{\mathfrak{m}}(K[\Delta])]_0 = \beta_{j-1}(\Delta)$ for all $j$ by Hochster's formula.

## 3. Level quotients

The goal of this section is to establish Theorem 1.2. Recall that an artinian graded $K$-algebra $A$ is a *level* ring if its socle is concentrated in one degree, that is, $[\operatorname{Soc} A]_j = 0$ if $j \neq e(A)$.

Let $\Delta$ be a simplicial complex on $[n]$. Recall that each subset $F \subseteq [n]$ induces the following simplicial subcomplexes of $\Delta$: the *star*

$$\operatorname{st}_\Delta F := \{G \in \Delta : F \cup G \in \Delta\},$$

and the *deletion*

$$\Delta_{-F} := \{G \in \Delta : F \cap G = \emptyset\}.$$

If $F \notin \Delta$, then $\mathrm{lk}_\Delta F = \mathrm{st}_\Delta F = \emptyset$.

Consider any vertex $k \in [n]$. Then $\mathrm{st}_\Delta k$ is the cone over the link $\mathrm{lk}_\Delta k$ with apex $k$. Hence its Stanley-Reisner ideal is $I_{\mathrm{st}_\Delta k} = I_\Delta : x_k$. Furthermore, the Stanley-Reisner ideal of the deletion $\Delta_{-k}$ considered as a complex on $[n]$ is $(x_k, J_{\Delta_{-k}}) = (x_k, I_\Delta)$, where $J_{\Delta_{-k}} \subset S$ is the extension ideal of the Stanley-Reisner ideal of $\Delta_{-k}$ considered as a complex on $[n] \setminus \{k\}$. Thus, we get the short exact sequence.

$$0 \to (K[\mathrm{st}_\Delta k])(-\deg x_k) \xrightarrow{x_k} K[\Delta] \to K[\Delta_{-k}] \to 0. \tag{1}$$

As preparation, we need the following technical result.

**Lemma 3.1.** *Let $\Delta$ be a $(d-1)$-dimensional Buchsbaum complex, and let $k$ be a vertex of $\Delta$ such that $\mathrm{lk}_\Delta k$ is 2-CM and $\Delta_k$ is Buchsbaum of dimension $d-1$. Then, for every linear system of parameters $\boldsymbol{\ell} := \ell_1, \dots, \ell_d$ of $K[\Delta]$, there is an exact sequence of graded modules*

$$0 \to (K[\mathrm{st}_\Delta k]/\boldsymbol{\ell})(-1) \to K[\Delta]/\boldsymbol{\ell} \to K[\Delta_{-k}]/\boldsymbol{\ell} \to 0,$$

*where the first map is induced by multiplication by $x_k$.*

*Proof.* For $j \in [d]$, set $\boldsymbol{\ell}_j := \ell_1, \dots, \ell_j$, so $\boldsymbol{\ell}_d = \boldsymbol{\ell}$. We will show that there is a short exact sequence

$$0 \to (K[\mathrm{st}_\Delta k]/\boldsymbol{\ell}_j)(-1) \to K[\Delta]/\boldsymbol{\ell}_j \to K[\Delta_{-k}]/\boldsymbol{\ell}_j \to 0. \tag{2}$$

Since $\Delta$ is Buchsbaum, the long exact cohomology sequence induced by Sequence (1) provides the exact sequence

$$0 \to H^{d-1}_{\mathfrak{m}}(K[\Delta]) \xrightarrow{\varphi} H^{d-1}_{\mathfrak{m}}(K[\Delta_{-k}]) \to H^{d}_{\mathfrak{m}}(K[\mathrm{st}_\Delta k])(-1)$$

Using that the modules $H^{d-1}_{\mathfrak{m}}(K[\Delta])$ and $H^{d-1}_{\mathfrak{m}}(K[\Delta_{-k}])$ are concentrated in degree zero, we see that every non-trivial element in $\operatorname{coker} \varphi$ gives a socle element of $H^{d}_{\mathfrak{m}}(K[\mathrm{st}_\Delta k])$. However, since $\mathrm{lk}_\Delta k$ is 2-CM, the socle of $H^{d}_{\mathfrak{m}}(K[\mathrm{st}_\Delta k])(-1)$ is concentrated in degree $d > 0$. We conclude that $\varphi$ is an isomorphism.

Since $\mathrm{st}_\Delta k$ is Cohen-Macaulay and has dimension $d-1$, it follows that

$$H^{i}_{\mathfrak{m}}(K[\Delta]) \cong H^{i}_{\mathfrak{m}}(K[\Delta_{-k}]) \quad \text{for all } i \neq d.$$

Using that $\Delta$ and $\Delta_{-k}$ are Buchsbaum this implies, for $j = 0, \dots, d-1$,

$$H^{0}_{\mathfrak{m}}(K[\Delta]/\boldsymbol{\ell}_j) \cong H^{0}_{\mathfrak{m}}(K[\Delta_{-k}]/\boldsymbol{\ell}_j). \tag{3}$$

We now show exactness of Sequence (2) by induction. Let $j \in \{0, \dots, d-1\}$. Using the induction hypothesis, multiplication by $\ell_{j+1}$ induces the following commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & 0 \arrow[d] & 0 :_{K[\Delta]/\boldsymbol{\ell}_j} \ell_{j+1} \arrow[d] & 0 :_{K[\Delta_{-k}]/\boldsymbol{\ell}_j} \ell_{j+1} \arrow[d] & \\
0 \arrow[r] & (K[\mathrm{st}_\Delta k]/\boldsymbol{\ell}_j)(-1) \arrow[r] \arrow[d, "\ell_{j+1}"] & K[\Delta]/\boldsymbol{\ell}_j \arrow[r] \arrow[d, "\ell_{j+1}"] & K[\Delta_{-k}]/\boldsymbol{\ell}_j \arrow[r] \arrow[d, "\ell_{j+1}"] & 0 \\
0 \arrow[r] & K[\mathrm{st}_\Delta k]/\boldsymbol{\ell}_j \arrow[r] & (K[\Delta]/\boldsymbol{\ell}_j)(1) \arrow[r] & (K[\Delta_{-k}]/\boldsymbol{\ell}_j)(1) \arrow[r] & 0
\end{tikzcd}$$

Since $K[\Delta]/\boldsymbol{\ell}_j$ and $K[\Delta_{-k}]/\boldsymbol{\ell}_j$ are Buchsbaum rings, we conclude that

$$0 :_{K[\Delta]/\boldsymbol{\ell}_j} \ell_{j+1} \cong H^0_{\mathfrak{m}}(K[\Delta]/\boldsymbol{\ell}_j) \cong H^0_{\mathfrak{m}}(K[\Delta_{-k}]/\boldsymbol{\ell}_j) \cong 0 :_{K[\Delta_{-k}]/\boldsymbol{\ell}_j} \ell_{j+1}.$$

Hence, the Snake lemma provides the exact sequence

$$0 \to (K[\operatorname{st}_\Delta k]/\boldsymbol{\ell}_{j+1})(-1) \to K[\Delta]/\boldsymbol{\ell}_{j+1} \to K[\Delta_{-k}]/\boldsymbol{\ell}_{j+1} \to 0,$$

as desired. □

Note that the above result remains true if one uses a system of parameters of arbitrary (positive) degrees. In the special case when $\Delta$ is an orientable homology manifold, Lemma 3.1 essentially reduces to [20], Proposition 4.24.

The following result is more general than Theorem 1.2. However, to take full advantage of this generality one needs an analogue of Theorem 1.1 for 2-Buchsbaum complexes.

**Theorem 3.2.** *Let $\Delta$ be a $(d-1)$-dimensional 2-Buchsbaum simplicial complex such that $\beta_{d-1}(\Delta) \neq 0$, and let $\boldsymbol{\ell} := \ell_1, \ldots, \ell_d$ be a linear system of parameters of $K[\Delta]$. Set $I := \bigoplus_{j=1}^{d-1} [\operatorname{Soc} K[\Delta]/\boldsymbol{\ell}]_j$. Then $(K[\Delta]/\boldsymbol{\ell})/I$ is a level ring of Cohen-Macaulay type $\beta_{d-1}(\Delta)$.*

*Proof.* Since we know that $\dim_K [K[\Delta]/\boldsymbol{\ell}]_d = \beta_{d-1}(\Delta)$ is it enough to show that the socle of $(K[\Delta]/\boldsymbol{\ell})/I$ vanishes in all degree $j < d$. By definition of $I$, this is true if $j = d-1$. Let $z \in K[\Delta]/\boldsymbol{\ell}$ be an element of degree $j \leq d-2$ such that $\mathfrak{m} \cdot z \subseteq I \subseteq \operatorname{Soc} K[\Delta]/\boldsymbol{\ell}$. We have to show that $z$ is already in $\operatorname{Soc} K[\Delta]/\boldsymbol{\ell}$.

To this end let $k \in [n]$ be any vertex of $\Delta$. Since $\Delta$ is 2-Buchsbaum, $\operatorname{lk}_\Delta k$ is 2-CM by Miyazaki ([11], Lemma 4.2). Thus, we may apply Lemma 3.1. We rewrite the exact sequence therein as

$$0 \to (K[\operatorname{st}_\Delta k]/\boldsymbol{\ell})(-1) \xrightarrow{\varphi} K[\Delta]/\boldsymbol{\ell} \to K[\Delta]/(\boldsymbol{\ell}, x_k) \to 0.$$

This sequence implies that there is some $y \in K[\operatorname{st}_\Delta k]/\boldsymbol{\ell}$ of degree $j$ such that $\varphi(y) = x_k \cdot z$. Since $x_k \cdot z$ is in $\operatorname{Soc} K[\Delta]/\boldsymbol{\ell}$, we conclude that $y$ is in the socle of $K[\operatorname{st}_\Delta k]/\boldsymbol{\ell})$. However, since $\operatorname{lk}_\Delta k$ is 2-CM of dimension $d-2$ and $\operatorname{st}_\Delta k$ is the cone over $\operatorname{lk}_\Delta k$, it follows that the socle of $K[\operatorname{st}_\Delta k]/\boldsymbol{\ell}$ is concentrated in degree $d-1 > j = \deg y$. This shows that $y = 0$, thus $x_k \cdot z = 0$. Since this is true for every vertex $k$, we have shown $\mathfrak{m} \cdot z = 0$, as required. □

## 4. Face enumeration

Now we discuss how Theorems 1.1 and 1.2 imply upper and lower bounds on the face vector of a Buchsbaum* complex $\Delta$.

The face or $f$-vector of a $(d-1)$-dimensional simplicial complex $\Delta$ is the sequence $f(\Delta) := (f_{-1}(\Delta), f_0(\Delta), \ldots, f_{d-1}(\Delta))$, where $f_j(\Delta)$ is the number of $j$-dimensional faces of $\Delta$. The same information is encoded in the $h$-vector $h(\Delta) := (h_0(\Delta), \ldots, h_d(\Delta))$, which is defined by

$$\frac{h_0(\Delta) + h_1(\Delta)t + \cdots + h_d(\Delta)t^d}{(1-t)^d} := \sum_{j \geq 0} \dim_K [K[\Delta]]_j t^j = \sum_{j=-1}^{d-1} \frac{f_j(\Delta) t^{j+1}}{(1-t)^{j+1}}.$$

More explicitly, this gives

$$h_j(\Delta) = \sum_{i=0}^{j} (-1)^{j-i} \binom{d-i}{j-i} f_{i-1}(\Delta) \quad \text{and} \quad f_{j-1}(\Delta) = \sum_{i=0}^{j} \binom{d-i}{j-i} h_i(\Delta).$$

The $h'$-vector $h'(\Delta) := (h'_0(\Delta), \ldots, h'_d(\Delta))$ is defined by

$$h'_j(\Delta) := h_j(\Delta) + \binom{d}{j} \sum_{i=0}^{j-1} (-1)^{j-i-1} \beta_{i-1}(\Delta).$$

If $\Delta$ is Buchsbaum, then its $h'$-vector is again a Hilbert function because, according to [15],

$$h'_j = \dim_K [K[\Delta]/\boldsymbol{\ell}]_j$$

if $\boldsymbol{\ell} := \ell_1, \ldots, \ell_d$ is a linear system of parameters of $K[\Delta]$.

Following [12], we define the $h''$-vector $h''(\Delta) := (h''_0(\Delta), \ldots, h''_d(\Delta))$ of $\Delta$ by

$$h''_j(\Delta) := h'_j(\Delta) - \binom{d}{j}\beta_{j-1}(\Delta) = h_j(\Delta) + \binom{d}{j} \sum_{i=0}^{j} (-1)^{j-i-1} \beta_{i-1}(\Delta) \quad \text{if } i < d$$

and

$$h''_d(\Delta) := \beta_{d-1}(\Delta).$$

The key for our purposes is that in case $\Delta$ is Buchsbaum* $h''(\Delta)$ also is a Hilbert function.

**Corollary 4.1.** *Let $\Delta$ be a $(d-1)$-dimensional Buchsbaum* complex. Using the notation of Theorem 1.2 set $\overline{K(\Delta)} := (K[\Delta]/\boldsymbol{\ell})/I$. Then $\overline{K(\Delta)}$ is a level algebra whose Hilbert function is given by*

$$\dim_K [\overline{K(\Delta)}]_j = h''_j(\Delta).$$

*Proof.* This follows by combining Theorem 1.1 and Theorem 1.2. $\square$

In order to use this information, we recall a result about Hilbert functions.

**Notation 4.2.** (i) We always use the following convention for binomial coefficients: If $a \in \mathbb{R}$ and $j \in \mathbb{Z}$ then

$$\binom{a}{j} := \begin{cases} \frac{a(a-1)\cdots(a-j+1)}{j!} & \text{if } j > 0 \\ 1 & \text{if } j = 0 \\ 0 & \text{if } j < 0. \end{cases}$$

(ii) Let $b > 0$ and $d \geq 0$ be integers. Then there are uniquely determined integers $m_s, \ldots, m_{d+1}$ such that $m_{d+1} \geq 0$, $n + d - 2 \geq m_d > m_{d-1} > \ldots > m_s \geq s \geq 1$, and

$$b = m_{d+1}\binom{n-1+d}{d} + \binom{m_d}{d} + \binom{m_{d-1}}{d-1} + \ldots + \binom{m_s}{s}.$$

This is called the *d-binomial expansion* of $b$. For any integer $j$ we set

$$b^{\langle d \rangle} := m_{d+1}\binom{n+d}{d+1} + \binom{m_d+1}{d+1} + \binom{m_{d-1}+1}{d} + \ldots + \binom{m_s+1}{s+1}.$$

(iii) If $b = 0$, then we put $b^{\langle d \rangle} := 0$ for all $d \in \mathbb{Z}$.

Assume $b > 0$. Note that then $s = d+1$ if and only if $\binom{n-1+d}{d}$ divides $b$. Furthermore, $\binom{m_d}{d} + \binom{m_{d-1}}{d-1} + \ldots + \binom{m_s}{s}$ is the standard $d$-binomial representation of $b - m_{d+1}\binom{n-1+d}{d}$.

We are ready to state a generalization of Macaulay's characterization of Hilbert functions of algebras to modules, which has been proven by Hulett [9] in characteristic zero and by Blancafort and Elias [3] in arbitrary characteristic.

**Theorem 4.3.** *For a numerical function $h : \mathbb{N}_0 \to \mathbb{N}_0$, the following conditions are equivalent:*

(a) *In non-negative degrees h is the Hilbert function of a module over* $S = K[x_1, \dots, x_n]$ *that is generated in degree zero, i.e., there is a graded S-module M whose minimal generators have degree zero such that* $h(j) = \dim_K [M]_j$ *whenever* $j \geq 0$.

(b) *For all integers* $j \geq 0$,

$$h(j+1) \leq h(j)^{\langle j \rangle}.$$

*Proof.* This follows from [3], Theorem 3.2. □

The following result provides restrictions on the face vectors of Buchsbaum* complexes with given Betti numbers. The upper bound strengthens [12], Theorem 4.3, for Buchsbaum complexes in the case of Buchsbaum* complexes.

**Theorem 4.4.** *Let* $\Delta$ *be a* $(d-1)$*-dimensional Buchsbaum* complex on n vertices. Then its* $h'$*-vector* $(h'_0, \dots, h'_d)$ *and* $h''$*-vector* $(h''_0, \dots, h''_d)$ *satisfy* $h' = h''_0 = 1$, $h'_1 = h''_1 = n - d$, $h''_d = \beta_{d-1}(\Delta)$, *and*

(a)

$$h'_{j+1} \leq \min \left\{ (h''_j)^{\langle j \rangle}, (h''_{j+2})^{\langle d-j-2 \rangle} + \beta_j(\Delta) \binom{d}{j+1} \right\} \quad \textit{if } \ 1 \leq j \leq d-2;$$

(b)

$$h''_{d-j} \geq \frac{h''_j}{\beta_{d-1}(\Delta)} \quad \textit{if } \ 1 \leq j \leq d-1.$$

*Proof.* (a) We use the notation of Corollary 4.1. The inequality $h'_{j+1} \leq (h''_j)^{\langle j \rangle}$ follows as in [12], Theorem 4.3, by applying Theorem 4.3 to the algebra $(K[\Delta]/\boldsymbol{\ell})/[I]_j S$.

The canonical module $\omega_{\overline{K(\Delta)}}$ of $\overline{K(\Delta)} = (K[\Delta]/\boldsymbol{\ell})/I$ is generated in degree $-d$ because $\overline{K(\Delta)}$ is level. Notice that, for all integers $j$,

$$\dim_K [\omega_{\overline{K(\Delta)}}]_{-j} = \dim_K [\overline{K(\Delta)}]_j = h''_j.$$

Hence Theorem 4.3 provides

$$h'_{j+1} - \beta_j(\Delta) \binom{d}{j+1} = h''_{j+1} \leq (h''_{j+2})^{\langle d-j-2 \rangle},$$

which completes the proof of (a).

(b) is a consequence of Theorem 2 in [18]. □

**Remark 4.5.** (i) It is a wide open problem to characterize the Hilbert functions of artinian level algebras. A systematic study of them was begun in [8].

(ii) The bound in Part (b) is never attained. This will be shown in the forthcoming paper [5].

(iii) According to Söderberg ([16], Theorem 4.7), the $h''$-vector also has to satisfy the determinantal condition

$$\begin{vmatrix} h''_{j-1} & h''_j & h''_{j+1} \\ r_{j-1} & r_j & r_{j+1} \\ r_{d-j+1} & r_{d-j} & r_{d-j-1} \end{vmatrix} \geq 0$$

for all integers $j$, where $r_j := \binom{n-1+j}{j}$. Söderberg's result depends on a joint conjecture with Boij in [4] that has been proven by Eisenbud and Schreyer [7].

Notice that Söderberg's condition characterizes the $h$-vectors of artinian level algebras up to multiplication by a rational number.

**Acknowledgement.** The author would like to thank Isabella Novik, Christos Athanasiadis, and an anonymous referee for helpful comments.

## References

[1] C. A. Athanasiadis, V. Welker, *Buchsbaum* complexes*, Preprint, 2009; also available at arXiv:0909.1931.

[2] E. Babson, I. Novik, *Face numbers and nongeneric initial ideals*, Electron. J. Comb. **11** (2006), Special volume in honor of R. Stanley, #R25, 23 pp.

[3] C. Blancafort, J. Elias, *On the growth of the Hilbert function of a module*, Math. Z. **234** (2000), 507–517.

[4] M. Boij, J. Söderberg, *Graded Betti numbers of Cohen-Macaulay modules and the multiplicity conjecture*, J. London Math. Soc. **78** (2008), 78–101.

[5] M. Boij, J. Migliore, R. Miró-Roig, U. Nagel, F. Zanello, in preparation.

[6] W. Bruns, J. Herzog, *Cohen-Macaulay rings. Rev. ed.*. Cambridge Studies in Advanced Mathematics **39**, Cambridge University Press, 1998.

[7] D. Eisenbud, F. Schreyer, *Betti numbers of graded modules and cohomology of vector bundles*, J. Amer. Math. Soc. **22** (2009), 859–888.

[8] A.V. Geramita, T. Harima, J. Migliore, and Y.S. Shin, *The Hilbert function of a level algebra*, Mem. Amer. Math. Soc. **186** (2007), No. 872.

[9] H. Hulett, *A generalization of Macaulay's theorem*, Comm. Algebra **23** (1995), 1249–1263.

[10] E. Miller, I. Novik, E. Swartz, *Face rings of simplicial complexes with singularities*, Preprint, 2010; also available at arXiv:1001.2812.

[11] M. Miyazaki, *On 2-Buchsbaum complexes*, J. Math. Kyoto Univ. **30** (1990), 367–392.

[12] I. Novik, E. Swartz, *Socles of Buchsbaum modules, complexes and posets*, Adv. M. **222** (2009), 2059–2084.

[13] I. Novik, E. Swartz, *Gorenstein rings through face rings of manifolds*, Compos. Math. **145** (2009), 993–1000.

[14] G. Reisner, *Cohen-Macaulay quotients of polynomial rings*, Adv. Math. **21** (1976), 30–49.

[15] P. Schenzel, *On the number of faces of simplicial complexes and the purity of Frobenius*, Math. Z. **178** (1981), 125–142.

[16] J. Söderberg, *Graded Betti numbers and h-vectors of level modules*, Preprint, 2006; also available at arXiv:0803.1645.

[17] R. P. Stanley, *Combinatorics and commutative algebra. Second edition.* Progress in Mathematics **41**, Birkhäuser Boston 1996.

[18] R. P. Stanley, *Cohen-Macaulay Complexes*, Higher Combinatorics, M. Aigner Ed., Reidel, Dordrecht and Boston (1977), 51–62.

[19] J. Stückrad, W. Vogel: *Buchsbaum rings and applications. An interaction between algebra, geometry and topology*, Springer-Verlag, Berlin, 1986.

[20] E. Swartz, *Face enumeration: From spheres to manifolds*, J. Eur. Math. Soc. **11** (2009), 449–485.

Department of Mathematics, University of Kentucky, 715 Patterson Office Tower, Lexington, KY 40506-0027, USA

*E-mail address*: `uwenagel@ms.uky.edu`